# Short Course on Experimental Dynamic Substructuring

## Module #8: Geometrically nonlinear systems




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## Outline

- **Geometric nonlinearities**
  - Bending-stretching coupling
  - Von-Karman beam/plate equation
  - Vibration modes for reduction?
- **Modal derivatives**
  - Definition
  - Dynamic modal derivatives
  - Static modal derivatives
- **Hurty-Craig-Bampton for geometric nonlinearities**
  - interface reduction
  - Modal derivatives of internal and interface modes
- **Flexible multibody systems**
  - Floating frame of reference
  - HCB vs Rubin
- **Summary**



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# Geometric Nonlinearities

*Definition*: geometrical nonlinearities are into play if the **change of geometry** of the system **under loading** cannot be neglected because **it influences the loading** (in amplitude and direction)

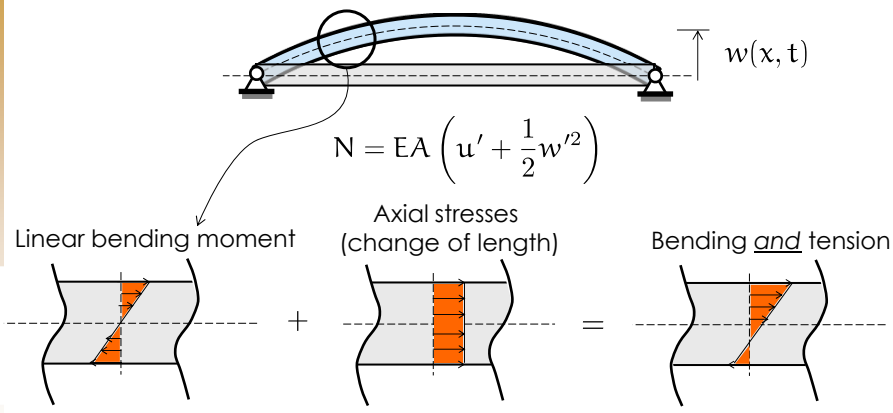
| | **Linear** | **Nonlinear** |
|---|---|---|
| Displacements | *Small* | *Large* |
| Rotations | *Small* | *Large* |
| Strains | *Small* | *Small/large* |

- ✓ *Multibody systems*
- ✓ *Thin-walled structures*





# String Effect




***Bending-stretching coupling***

O. Thomas, A. Sénéchal, J.-F. Deü, ***Hardening/softening behavior and reduced order modeling of nonlinear vibrations of rotating cantilever beams***, *Nonlinear Dynamics*, 2016.

## von Karman beam FE equation

In/out of plane coupling via nonlinear terms:

$$\begin{bmatrix} \mathbf{M}_{ww} & \mathbf{0} \\ 0 & \mathbf{M}_{\mathrm{uu}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{w}} \\ \ddot{\mathbf{u}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ww} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\mathrm{uu}} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} {}_2\mathbf{f}_w(\mathbf{w},\mathbf{u}) \\ {}_2\mathbf{f}_{\mathrm{u}}(\mathbf{w},\mathbf{w}) \end{bmatrix} + \begin{bmatrix} {}_3\mathbf{f}_w(\mathbf{w},\mathbf{w},\mathbf{w}) \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_w \\ \mathbf{0} \end{bmatrix}$$

- Linearly decoupled (beam and bar)
- This particular structure of the nonlinear terms only due to the symmetry of the system
- For thin-walled structures, still one can distinguish between **slow, bending** dominated dynamics and **fast, axial** dominated dynamics; this is a key ingredient for model order reduction



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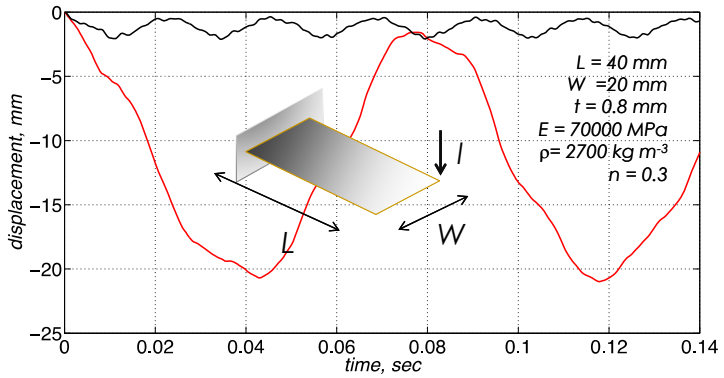
## Vibration modes?

$$\left[\mathbf{K} - \omega_{\mathrm{i}}^2 \mathbf{M}\right] \boldsymbol{\phi}_{\mathrm{i}} = \mathbf{0} \qquad \boldsymbol{\Phi} = [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{\mathrm{m}}]$$




P. Tiso, *Optimal second order reduction basis selection for nonlinear transient analysis*, IMAC-XXIX, 2011.



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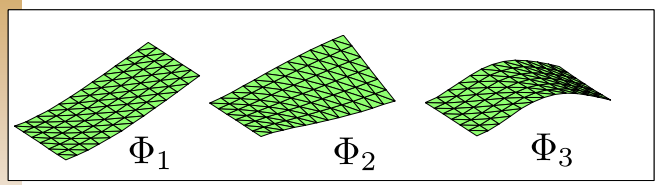
## What is missing?

Slow vibration modes only contain out-of-plane displacements:




$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}_w \\ \boldsymbol{\Phi}_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_w \\ \mathbf{0}_u \end{bmatrix}$$

Overly-stiff behavior!

$$\begin{bmatrix} \mathbf{M}_{ww} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{uu} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{w}} \\ \ddot{\mathbf{u}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ww} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{uu} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} {}_2\mathbf{f}_w(\mathbf{w},\mathbf{u}) \\ {}_2\mathbf{f}_u(\mathbf{w},\mathbf{w}) \end{bmatrix} + \begin{bmatrix} {}_3\mathbf{f}_w(\mathbf{w},\mathbf{w},\mathbf{w}) \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_w \\ \mathbf{0} \end{bmatrix}$$

Need appropriate in-plane contributions. Seeking them between the set of high-frequency eigenmodes is not a good option:

1. Which ones?
2. Need to extract a lot of them. For large systems, this is expensive.



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## Modal Derivatives - definition

For geometrically nonlinear structural dynamics, slow vibration modes are still the dominant ingredient of the response

$$(\mathbf{K}|_{eq} - \omega_i^2 \mathbf{M})\boldsymbol{\phi}_i = \mathbf{0} \qquad \mathbf{u}(t) \approx \sum_{i=1}^{m} \boldsymbol{\phi}_i \eta_i(t) = \boldsymbol{\Phi}\mathbf{q}(t),$$

Then, we need to know how modes change under underlying finite displacements.

Modal derivative

$$(\mathbf{K}|_{eq} - \omega_i^2|_{eq}\mathbf{M}) \left.\frac{\partial \boldsymbol{\phi}_i}{\partial q_j}\right|_{eq} + \left( \left.\frac{\partial \mathbf{K}}{\partial q_j}\right|_{eq} - \left.\frac{\partial \omega_i^2}{\partial q_j}\right|_{eq} \mathbf{M} \right) \boldsymbol{\phi}_i|_{eq} = \mathbf{0}$$

The *directional derivative* of the stiffness matrix is defined as

$$\left.\frac{\partial \mathbf{K}}{\partial q_j}\right|_{eq} = \left.\frac{\partial \mathbf{K}(\mathbf{u} = \eta_j \boldsymbol{\phi}_j|_{eq}))}{\partial q_j}\right|_{q_j=0}$$



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## Modal derivatives - calculation

Recall the orthogonality property of vibration modes

$$\boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i = 1 \qquad \forall i \in \{1, 2, \ldots, m\}.$$

Differentiating and accounting for symmetry of **M**

$$\boldsymbol{\phi}_i^T \mathbf{M} \frac{\partial \boldsymbol{\phi}_i}{\partial q_j} + \boldsymbol{\phi}_i^T \mathbf{M}^T \frac{\partial \boldsymbol{\phi}_i}{\partial q_j} = 0 \qquad \forall i, j \in \{1, 2, \ldots, m\}$$

$$\boldsymbol{\phi}_i^T|_{eq} \mathbf{M} \left. \frac{\partial \boldsymbol{\phi}_i}{\partial q_j} \right|_{eq} = 0 \qquad \forall i, j \in \{1, 2, \ldots, m\}$$

This gives an extra condition to close the system: direct method

$$\begin{bmatrix} [\mathbf{K}|_{eq} - \omega_i^2|_{eq} \mathbf{M}]_{n \times n} & -[\mathbf{M}\boldsymbol{\phi}_i|_{eq}]_{n \times 1} \\ -[\mathbf{M}\boldsymbol{\phi}_i|_{eq}]_{1 \times n}^T & 0_{1 \times 1} \end{bmatrix} \begin{bmatrix} [1] \left. \frac{\partial \boldsymbol{\phi}_i}{\partial q_j} \right|_{eq} \\ \left. \frac{\partial \omega_i^2}{\partial q_j} \right|_{eq} \end{bmatrix} = \begin{bmatrix} -\left. \frac{\partial \mathbf{K}}{\partial q_j} \right|_{eq} \boldsymbol{\phi}_i|_{eq} \\ 0 \end{bmatrix}.$$



## Static Modal Derivatives

Problem with the definition: it is questionable if *modal amplitudes* (i.e. structural displacements) can be treated as *parameters*. Does sensitivity make sense?

It was early proposed [1] to "*neglect the mass matrix*", which leads to the *Static Modal Derivatives* (SMDs)

$$\mathbf{K}|_{eq} \left. \frac{\partial \boldsymbol{\phi}_i}{\partial q_j} \right|_{eq}^{s} = -\left. \frac{\partial \mathbf{K}}{\partial q_j} \right|_{eq} \boldsymbol{\phi}_i|_{eq}$$

Bring a lot of advantages:

- ✓ Easier to compute: set of linear problems with same **K**
- ✓ More consistent with the physics (more on this later)
- ✓ Symmetric (m*(m+1)/2 SMDs)

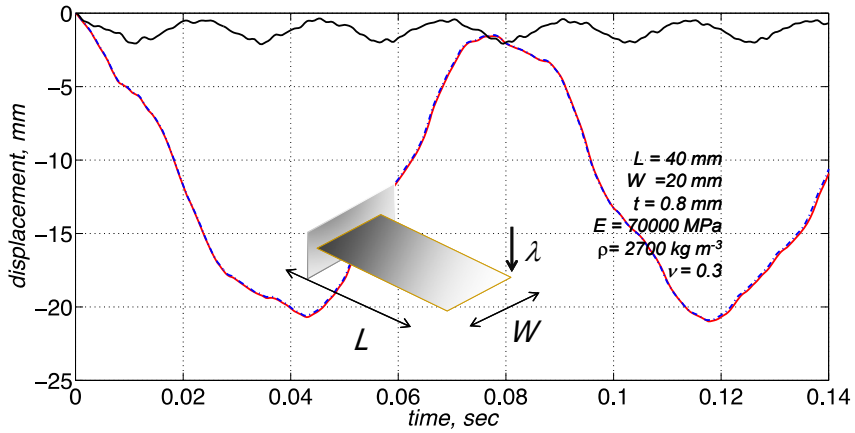
$$\left. \frac{\partial \boldsymbol{\phi}_i}{\partial q_j} \right|_{eq}^{s} = \left. \frac{\partial \boldsymbol{\phi}_j}{\partial q_i} \right|_{eq}^{s}$$

[1] Idelsohn, S. R., Cardona, A. ***A reduction method for nonlinear structural dynamic analysis***, CMAME (1985)

## Modal Derivatives - effectiveness

Basis enrichment

$$\Phi = [\phi_1, \ldots, \phi_m, \ldots, \theta_{ij}, \ldots]$$



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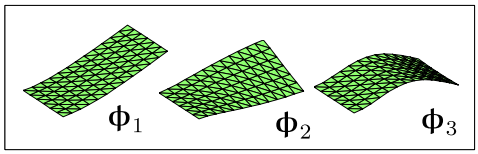
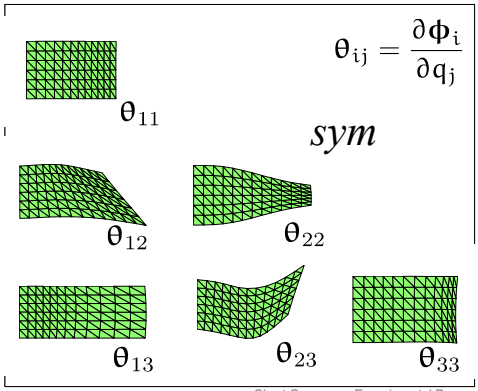
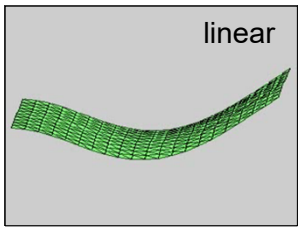
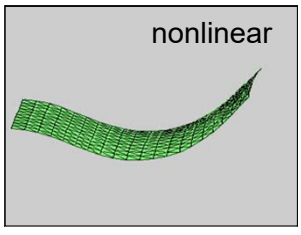
## Physical Interpretation




$$\theta_{ij} = \frac{\partial \phi_i}{\partial q_j}$$



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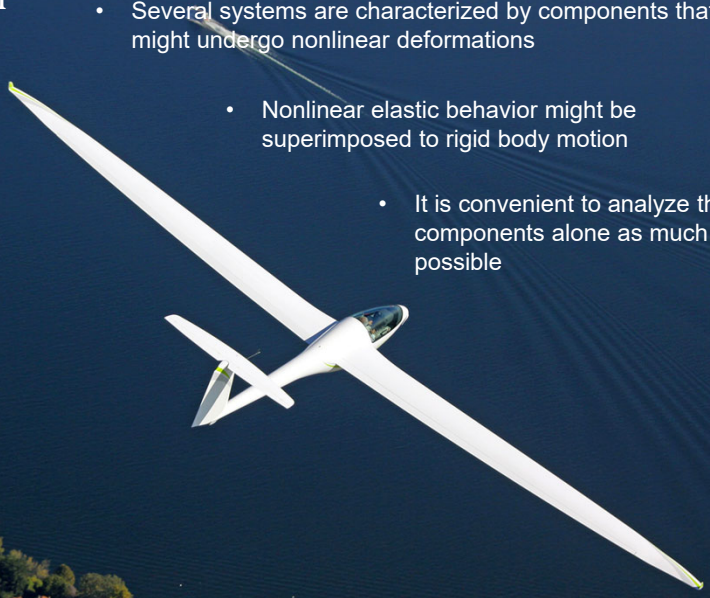
# Motivation

- Several systems are characterized by components that might undergo nonlinear deformations
  - Nonlinear elastic behavior might be superimposed to rigid body motion
    - It is convenient to analyze the components alone as much as possible



Based on:

L. Wu, P. Tiso, ***A Modal Derivatives Enhanced Craig-Bampton Method for Geometrically Nonlinear Structures***, preprint

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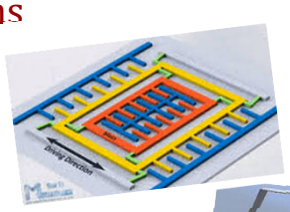
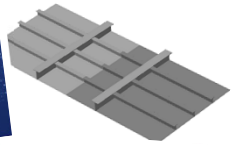
# Possible Applications







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## Theory

*Each substructure is assumed geometrically nonlinear*

$$\mathbf{M}_{(s)}\ddot{\mathbf{u}}_{(s)}\{t\} + \mathbf{f}_{(s)}\left\{\mathbf{u}_{(s)}\{t\}\right\} = \mathbf{g}_{(s)}\{t\} + \mathbf{p}_{(s)}\{t\}, \quad s = 1, \ldots, H$$

*Linearize:*

$$\begin{bmatrix} \mathbf{M}_{(s)}^{BB} & \mathbf{M}_{(s)}^{BI} \\ \mathbf{M}_{(s)}^{IB} & \mathbf{M}_{(s)}^{II} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_{(s)}^{B} \\ \ddot{\mathbf{q}}_{(s)}^{I} \end{bmatrix} + \begin{bmatrix} \overline{\mathbf{K}}_{(s)}^{BB} & \overline{\mathbf{K}}_{(s)}^{BI} \\ \overline{\mathbf{K}}_{(s)}^{IB} & \overline{\mathbf{K}}_{(s)}^{II} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{(s)}^{B} \\ \mathbf{q}_{(s)}^{I} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{(s)}^{B} \\ \mathbf{g}_{(s)}^{I} \end{bmatrix} + \begin{bmatrix} \mathbf{p}_{(s)}^{B} \\ \mathbf{0} \end{bmatrix}$$

*and apply Hurty-Craig-Bampton:*

$$\mathbf{q}_{(s)} = \begin{bmatrix} \mathbf{q}_{(s)}^{B} \\ \mathbf{q}_{(s)}^{I} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{(s)}^{BB} & \mathbf{0} \\ \boldsymbol{\Psi}_{(s)}^{IB} & \boldsymbol{\Phi}_{(s)}^{II} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{(s)}^{B} \\ \boldsymbol{\eta}_{(s)}^{I} \end{bmatrix} \triangleq \mathbf{X}_{(s)}\boldsymbol{\gamma}_{(s)}$$



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## Theory

After primal assembly, the linearized system looks like

$$\begin{bmatrix} \widetilde{\mathbf{M}}_{\mathcal{CB}}^{BB} & \widetilde{\mathbf{M}}_{\mathcal{CB}}^{BI} \\ \widetilde{\mathbf{M}}_{\mathcal{CB}}^{IB} & \mathbf{I}_{\mathcal{CB}}^{II} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_{\mathcal{CB}}^{B} \\ \ddot{\boldsymbol{\eta}}_{\mathcal{CB}}^{I} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{K}}_{\mathcal{CB}}^{BB} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_{\mathcal{CB}}^{2} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{\mathcal{CB}}^{B} \\ \boldsymbol{\eta}_{\mathcal{CB}}^{I} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{g}}_{\mathcal{CB}}^{B} \\ \widetilde{\mathbf{g}}_{\mathcal{CB}}^{I} \end{bmatrix}$$

If many components are present and/or the interface is large, the ROM is still large. We can then apply *interface reduction*

$$\widetilde{\mathbf{M}}_{\mathcal{CB}}^{BB}\ddot{\mathbf{q}}_{\mathcal{CB}}^{B} + \widetilde{\mathbf{K}}_{\mathcal{CB}}^{BB}\mathbf{q}_{\mathcal{CB}}^{B} = \widetilde{\mathbf{g}}_{\mathcal{CB}}^{B}$$

By extracting dominant modes on the interface dofs only (internal dynamics is statically condensed)

$$\left(\widetilde{\mathbf{K}}_{\mathcal{CB}}^{BB} - \widetilde{\omega}_j^2\widetilde{\mathbf{M}}_{\mathcal{CB}}^{BB}\right)\widetilde{\boldsymbol{\phi}}_j^{BB} = \mathbf{0}, \quad j = 1, \ldots, n_{\mathcal{CB}}^{B}$$



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# Theory

The interface is then reduced as:

$$\boldsymbol{\xi}_{\mathcal{CB}} = \begin{bmatrix} \mathbf{q}^B_{\mathcal{CB}} \\ \boldsymbol{\eta}^I_{\mathcal{CB}} \end{bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{\Phi}}^{BB} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}^B_{\mathcal{CC}} \\ \boldsymbol{\eta}^I_{\mathcal{CB}} \end{bmatrix} \triangleq \mathbf{X}_{\mathcal{CC}} \boldsymbol{\xi}_{\mathcal{CC}}$$

For two substructures, the final reduction basis looks like

$$\mathbf{q}_{\mathcal{G}} = \begin{bmatrix} \mathbf{q}^B_{(1)} \\ \mathbf{q}^I_{(1)} \\ \mathbf{q}^B_{(2)} \\ \mathbf{q}^I_{(2)} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I}^{BB}_{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\Psi}^{IB}_{(1)} & \boldsymbol{\Phi}^{II}_{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}^{BB}_{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Psi}^{IB}_{(2)} & \boldsymbol{\Phi}^{II}_{(2)} \end{bmatrix}}_{\mathbf{X}_{\mathcal{G}}} \underbrace{\begin{bmatrix} \mathbf{I}^{BB}_{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}^{II}_{(1)} & \mathbf{0} \\ \mathbf{I}^{BB}_{(2)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}^{II}_{(2)} \end{bmatrix}}_{\mathbf{L}_{\mathcal{CB}}} \underbrace{\begin{bmatrix} \widetilde{\boldsymbol{\Phi}}^{BB} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}^{II}_{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}^{II}_{(2)} \end{bmatrix}}_{\mathbf{X}_{\mathcal{CC}}} \begin{bmatrix} \boldsymbol{\eta}^B_{\mathcal{CC}} \\ \boldsymbol{\eta}^I_{(1)} \\ \boldsymbol{\eta}^I_{(2)} \end{bmatrix} =$$

$$\begin{bmatrix} \widetilde{\boldsymbol{\Phi}}^{BB}_{(1)} & \mathbf{0} & \mathbf{0} \\ \widetilde{\boldsymbol{\Phi}}^{IB}_{(1)} & \boldsymbol{\Phi}^{II}_{(1)} & \mathbf{0} \\ \widetilde{\boldsymbol{\Phi}}^{BB}_{(2)} & \mathbf{0} & \mathbf{0} \\ \widetilde{\boldsymbol{\Phi}}^{IB}_{(2)} & \mathbf{0} & \boldsymbol{\Phi}^{II}_{(1)} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}^B_{\mathcal{CC}} \\ \boldsymbol{\eta}^I_{(1)} \\ \boldsymbol{\eta}^I_{(2)} \end{bmatrix}$$



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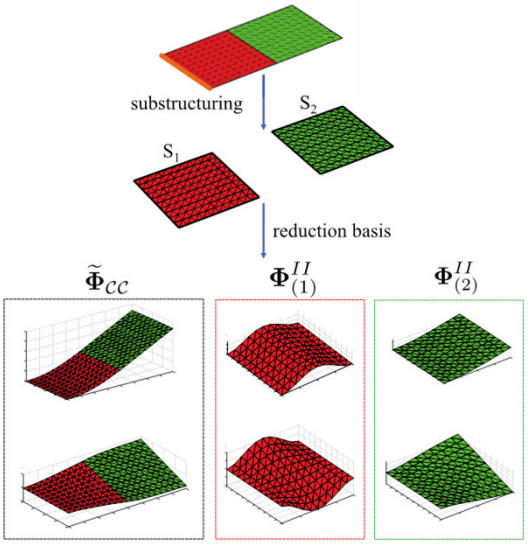
# Linear HCB + Interface reduction



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## Enhancement with modal derivatives

It is now possible to compute MDs for

- ✓ The internal VMs of each substructure

$$\boldsymbol{\theta}^{II}_{jl,(s)} \triangleq \left.\frac{\partial^2 \mathbf{u}^I_{(s)}}{\partial y_{l,(s)} \partial y_{j,(s)}}\right|_0 = -\left(\overline{\mathbf{K}}^{II}_{(s)}\right)^{-1} \left.\frac{\partial \mathbf{K}^{II}_{(s)}}{\partial y_{l,(s)}}\right|_0 \boldsymbol{\phi}^{II}_{j,(s)}$$

$$\left[\boldsymbol{\phi}^{II}_{1,(s)} \quad \cdots \quad \boldsymbol{\phi}^{II}_{m^I_{(s)},(s)} \quad \cdots \quad \boldsymbol{\theta}^{II}_{jl,(s)} \quad \cdots\right] \triangleq \left[\boldsymbol{\Phi}^{II}_{(s)} \quad \boldsymbol{\Theta}^{II}_{(s)}\right]$$

- ✓ The Characteristic Constraint Modes (CCMs) (analogous formula)

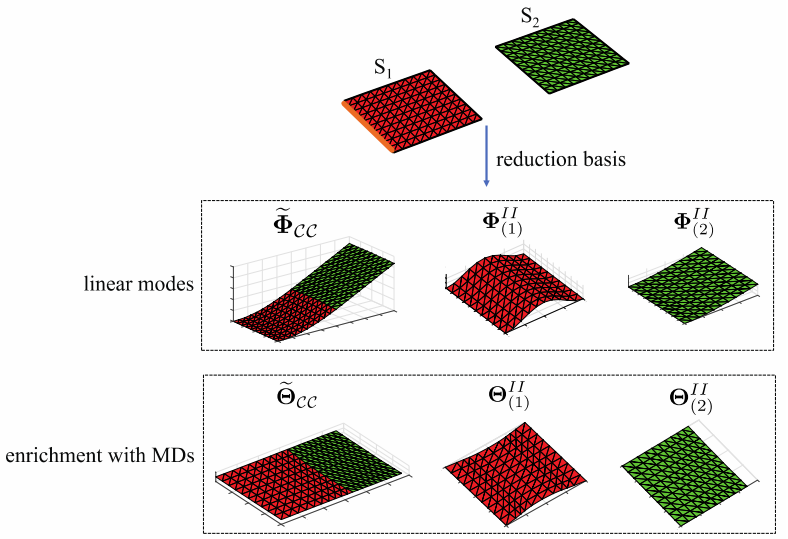
The internal and interface reduction is now enriched with MDs

$$\mathbf{u}^I_{(s)} = \begin{bmatrix}\boldsymbol{\Phi}^{II}_{(s)} & \boldsymbol{\Theta}^{II}_{(s)}\end{bmatrix} \begin{bmatrix}\boldsymbol{\eta}^I_{(s)} \\ \boldsymbol{\zeta}^I_{(s)}\end{bmatrix} \quad \mathbf{u}_{\mathcal{C}} = \begin{bmatrix}\widetilde{\boldsymbol{\Phi}}_{\mathcal{C}\mathcal{C}} & \widetilde{\boldsymbol{\Theta}}_{\mathcal{C}\mathcal{C}}\end{bmatrix} \begin{bmatrix}\boldsymbol{\eta}^B_{\mathcal{C}\mathcal{C}} \\ \boldsymbol{\zeta}^B_{\mathcal{C}\mathcal{C}}\end{bmatrix}$$





## Enhancement with modal derivatives



# Reduced forces

Due to the polynomial form of von-Karman kinematics, the internal forces can be expressed directly in modal coordinates

$$\widetilde{\mathbf{f}}_{(s)} = {}^{2}\widetilde{\mathbf{Q}}_{(s)}\boldsymbol{\xi}_{(s)} + \left({}^{3}\widetilde{\mathbf{Q}}_{(s)} \cdot \boldsymbol{\xi}_{(s)}\right)\boldsymbol{\xi}_{(s)} + \left[\left({}^{4}\widetilde{\mathbf{Q}}_{(s)} \cdot \boldsymbol{\xi}_{(s)}\right) \cdot \boldsymbol{\xi}_{(s)}\right]\boldsymbol{\xi}_{(s)}$$

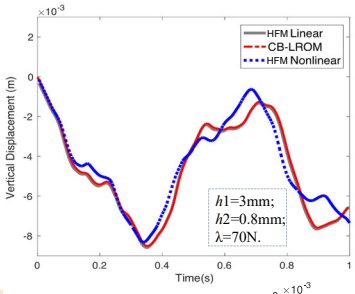
Computed offline once modes are known

The tangent stiffness matrix is likewise obtained.

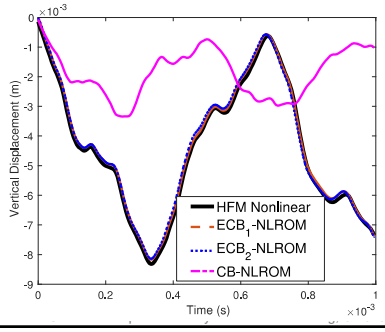




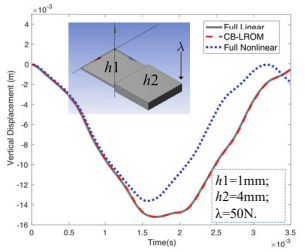
# Thick-thin plate



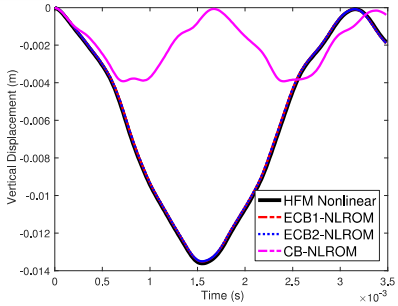

| Reduction Basis | $S_1$ | $S_2$ | $B$ | |
|---|---|---|---|---|
| CB-LROM | number of IVMs<br>5 | 10 | number of SCC modes<br>10 | total DoFs<br>25 |
| $ECB_1$-NLROM | number of IVMs<br>5<br>number of MDs<br>10 | 10<br>20 | number of SCC modes<br>10<br>number of MDs<br>10 | total DoFs<br><br>65 |
| $ECB_2$-NLROM | number of IVMs<br>5<br>number of MDs<br>0 | 10<br>20 | number of SCC modes<br>10<br>number of MDs<br>0 | total DoFs<br><br>45 |
| CB-NLROM | number of IVMs<br>120 | 120 | number of SCC modes<br>66 | total DoFs<br>306 |



# Thin-thick plate




| Reduction Basis | $S_1$ | $S_2$ | $B$ | |
|---|---|---|---|---|
| CB-LROM | number of IVMs | | number of SCC modes | total DoFs |
| | 10 | 5 | 10 | 25 |
| ECB$_1$-NLROM | number of IVMs | | number of SCC modes | total DoFs |
| | 10 | 5 | 10 | |
| | number of MDs | | number of MDs | 66 |
| | 20 | 6 | 15 | |
| ECB$_2$-NLROM | number of IVMs | | number of SCC modes | total DoFs |
| | 10 | 5 | 10 | |
| | number of MDs | | number of MDs | 60 |
| | 20 | 0 | 15 | |
| CB-NLROM | number of IVMs | | number of SCC modes | total DoFs |
| | 120 | 120 | 66 | 306 |



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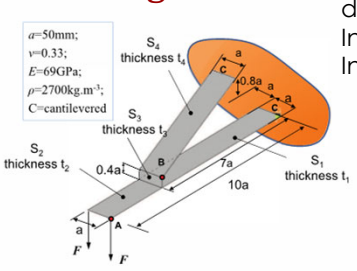
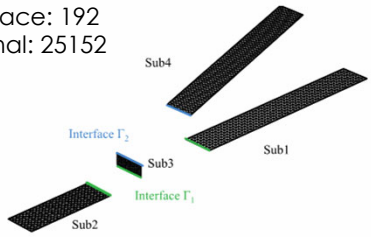
# Joined Wing

dof: 25344
Interface: 192
Internal: 25152




Sketch of the joined wing model




FE model of substructures

| Methods | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $B$ | |
|---|---|---|---|---|---|---|
| CB-LROM | number of IVMs | | | | number of SCC modes | total DoFs |
| | 10 | 10 | 10 | 10 | 10 | 50 |
| ECB-NLROM | number of IVMs | | | | number of SCC modes | total DoFs |
| | 10 | 10 | 10 | 10 | 10 | |
| | number of MDs | | | | number of MDs | 125 |
| | 15 | 15 | 15 | 15 | 15 | |
| CB-NLROM | number of IVMs | | | | number of CMs | total DoFs |
| | 50 | 50 | 50 | 50 | 192 | 392 |



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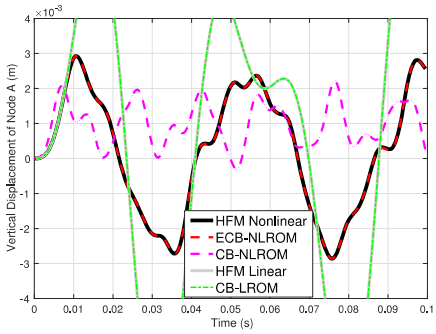
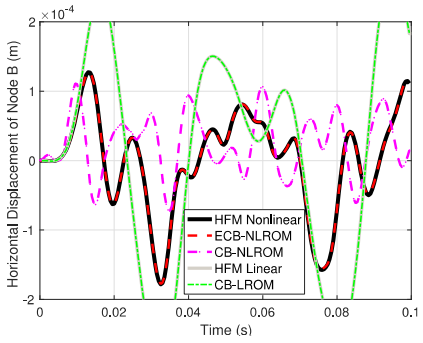
# Joined Wing




| Numerical models | CPU time $t_{full}(s)$ | $t_{red}(s)$ | speed up factor $S_u$ | Global relative error $GRE_m(\%)$ |
|---|---|---|---|---|
| Model-I(1) | 65.36 | 5.78 | 11.31 | 1.52 |
| Model-I(2) | 70.59 | 7.23 | 9.76 | 1.31 |
| Model-II | 1194 | 16.65 | 71.71 | 0.54 |



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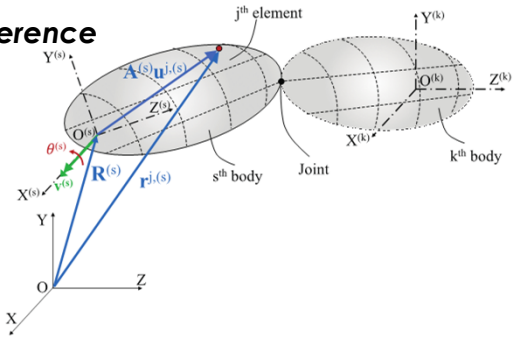
# Flexible Multibody Systems

***Floating frame of reference***




$$\mathbf{r}^{j,(s)} = \mathbf{R}^{(s)} + \mathbf{A}^{(s)}\mathbf{u}^{j,(s)} = \mathbf{R}^{(s)} + \mathbf{A}^{(s)}\mathbf{N}^{j,(s)}\left(\mathbf{q}_0^{j,(s)} + \mathbf{q}_f^{j,(s)}\right)$$

- ✓ *Each component referred to its own frame*
- ✓ *Allows for substructuring and independent reduction*
- ✓ *Rigid and flexible joints possible*



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# Multibody EoMs

Configuration-dependent mass matrix

Floating frame dofs

Constraint forces

$$\begin{bmatrix} \mathbf{M}_{\mathrm{RR}}^{(s)} & \mathbf{M}_{\mathrm{R}\theta}^{(s)} & \mathbf{M}_{\mathrm{Rm}}^{(s)} \\ & \mathbf{M}_{\theta\theta}^{(s)} & \mathbf{M}_{\theta\mathrm{m}}^{(s)} \\ \mathrm{sym} & & \mathbf{M}_{\mathrm{mm}}^{(s)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{R}}^{(s)} \\ \ddot{\boldsymbol{\theta}}^{(s)} \\ \ddot{\mathbf{q}}_{\mathrm{m}}^{(s)} \end{bmatrix} - \begin{bmatrix} \mathbf{Q}_{\mathrm{R}}^{(s)} \\ \mathbf{Q}_{\theta}^{(s)} \\ \mathbf{Q}_{\mathrm{m}}^{(s)} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{f}_{\mathrm{m}}^{(s)} \end{bmatrix} + \begin{bmatrix} (\mathbf{C}_{\mathrm{R}}^{(s)})^{\mathrm{T}} \\ (\mathbf{C}_{\theta}^{(s)})^{\mathrm{T}} \\ (\mathbf{C}_{\mathrm{m}}^{(s)})^{\mathrm{T}} \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} \mathbf{g}_{\mathrm{R}}^{(s)} \\ \mathbf{g}_{\theta}^{(s)} \\ \mathbf{g}_{\mathrm{m}}^{(s)} \end{bmatrix}$$

$$\mathbf{C}\{\mathbf{R}, \boldsymbol{\theta}, \mathbf{q}_{\mathrm{m},\nu}\} = \mathbf{0},$$

Elastic dofs

Nonlinear elastic forces

Gen. ext. load

Constraints

Inertial forces

Achieve reduction by **Component Mode Synthesis**



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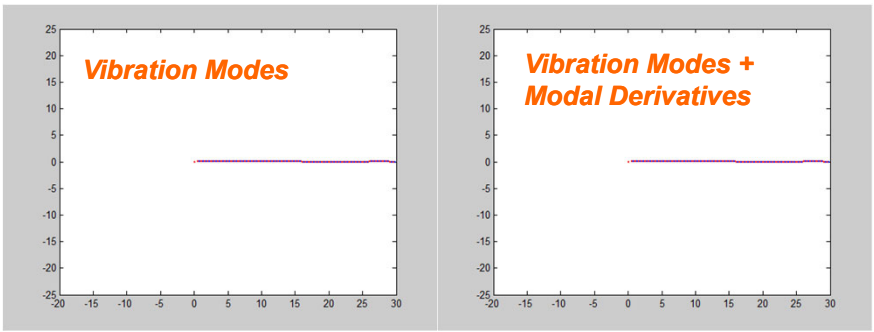
# Flexible Multibody Systems

Dr. Long Wu




L. Wu, P. Tiso, ***Nonlinear model order reduction for flexible multibody dynamics: a modal derivatives approach***, Multibody System Dynamics, 2016

- ✓ Hurty-Craig-Bampton reduction per component
- ✓ nodal fixed frame
- ✓ Single node interface

Paolo Tiso



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## Floating Frame Definition

Accuracy of approximate kinematics models (i.e. von Karman) improves if *relative displacements are minimized*

***Mean Axis***: dynamically define the FFR position and orientation by minimizing the relative kinetic energy.

*Relative kinetic energy:*

$$\mathcal{T}_r = \sum_j \frac{1}{2} \int_{V^j} \rho^j \left[\dot{\mathbf{r}}^j - \dot{\mathbf{R}} - \mathcal{B}^j \dot{\theta}^j\right]^T \left[\dot{\mathbf{r}}^j - \dot{\mathbf{R}} - \mathcal{B}^j \dot{\theta}^j\right] dV^j$$

*Minimize w.r.t. the frame position and orientation:*

$$\frac{\partial \mathcal{T}_r}{\partial \dot{\mathbf{R}}} = \mathbf{0} \qquad \frac{\partial \mathcal{T}_r}{\partial \boldsymbol{\omega}} = \mathbf{0} \qquad \boldsymbol{\omega} = 2\mathbf{E}\dot{\theta}$$

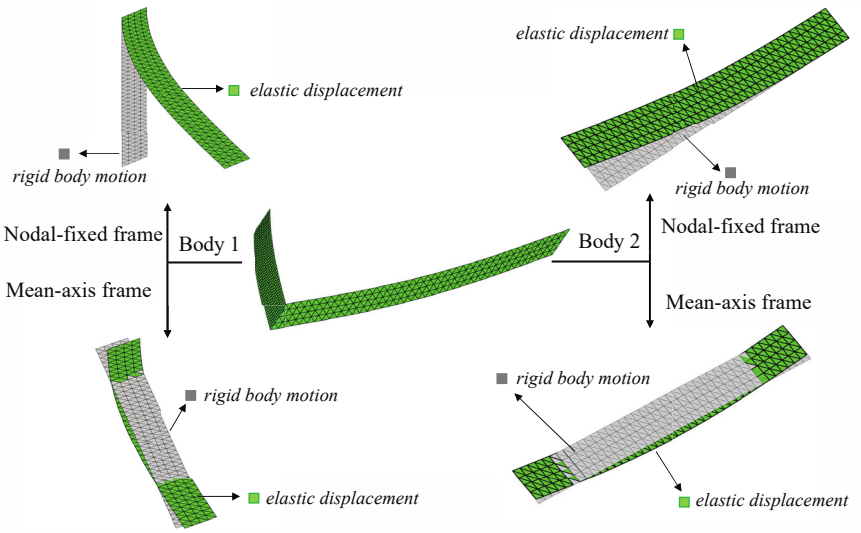
*Linearize the constraint and integrate in time*

$$\mathbf{S}\dot{\mathbf{q}}_f = \mathbf{0} \rightarrow \mathbf{S}\mathbf{q}_f = \mathbf{0}$$



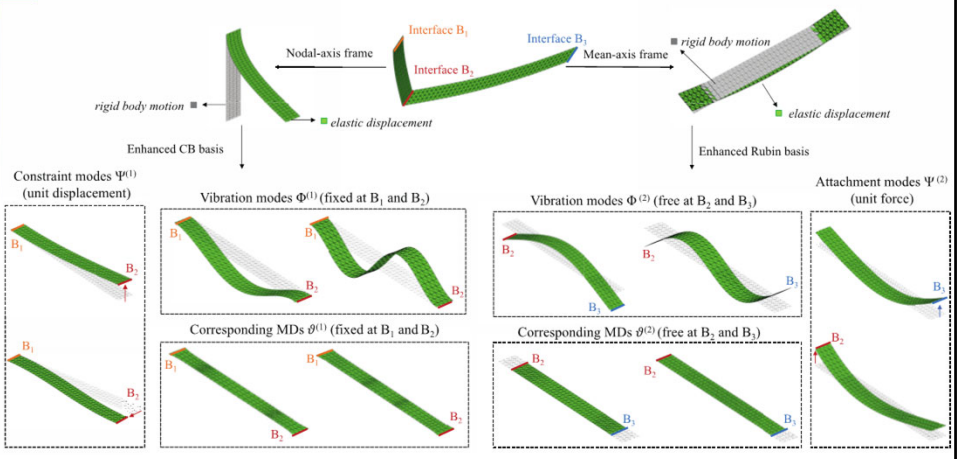


## Nodal Fixed vs Mean Axis



# Component reduction basis

- Use of the Rubin Method + modal derivatives



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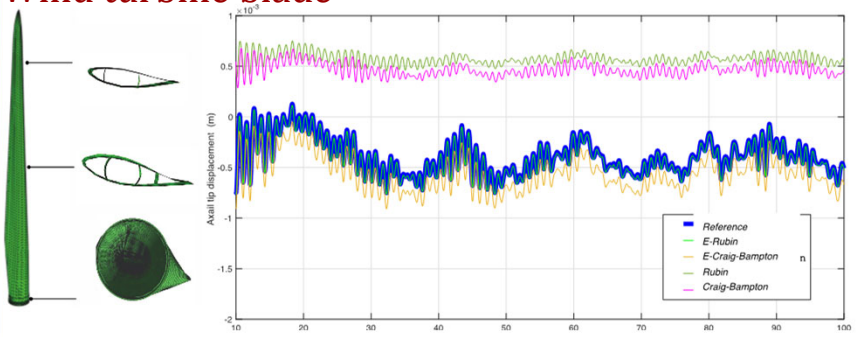
# Wind turbine blade




| floating frame | HFM | ROM offline | | online | Number of iterations | speed up factor | |
|---|---|---|---|---|---|---|---|
| | $t_{full}$ | $t_{off1}$ | $t_{off2}$ | $t_{on}$ | $N$ | $S_1$ | $S_2$ |
| mean-axis | 155307$s$ | 2053s | 230s | 72s | 10598 | 2157 | 65 |
| nodal-fixed | 49254s | 13.25s | 241s | 125s | 17758 | 394 | 129.87 |

Rubin + mean axis FFR outperforms Craig-Bampton + nodal fixed FFR

L.Wu, P. Tiso, K. A. van Keulen, K. Tatsis, E. Chatzi, ***A modal derivatives enhanced Rubin substructuring method for geometrically nonlinear multibody systems,*** Multibody System Dynamics, 2019



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## Summary

- **MDs can be applied in a substructuring context**, both on internal and interface modes
- They greatly improve the solution as compared to several high-frequency modes
- Role of MDs in reproducing **the rigid body modes** in highlighted
- In a floating frame of reference, the most promising combination seems mean axis definition + Rubin + MDs. This **minimizes the elastic deviation** from reference frame (and thus helps staying within the validity of the kinematic model) and **more naturally accounts for the rigid body motion**



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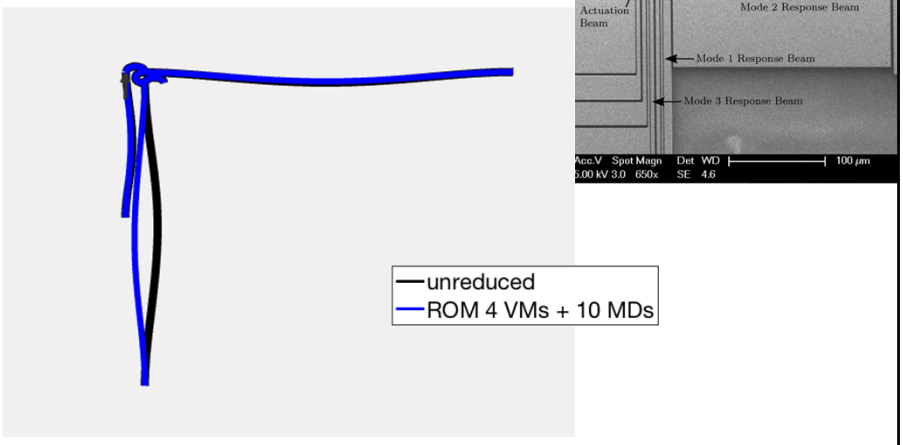
## At this IMAC

*Nonlinear substructuring for frequency dividers (parametric resonance)*







***A Component Mode Synthesis Method for the Nonlinear Dynamics Analysis of Frequency Dividers***
#7602 | J. van der Broek–TU Delft; J. Marconi–Politecnico di Milano; P. Tiso–ETH Zürich; F. Alijani–Delft University of Technology



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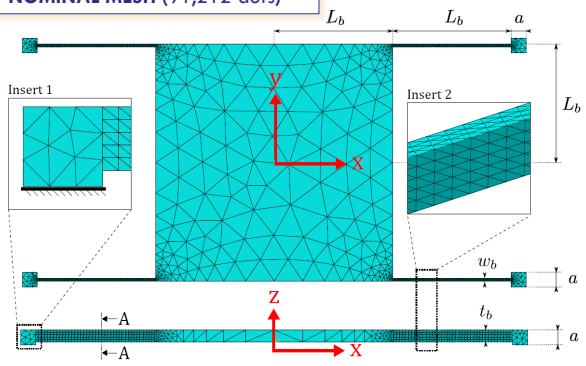
# At this IMAC

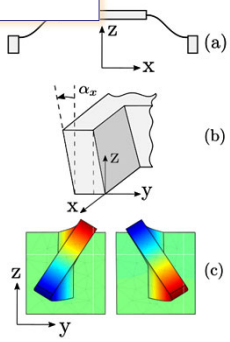
*Geometrically nonlinear ROM with embedded parametrized defects.*

**NOMINAL MESH** (91,212 dofs)

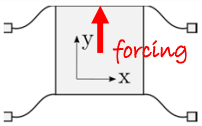



(a) second VM, (b) wall angle, (c) twist of the beams

**3 DEFECTS**




**ACTUATION MODE ($\phi_1$)**




***A Nonlinear Reduced Order Model with Parametrized Shape Defects*** #7613 | J. Marconi– Politecnico di Milano; P. Tiso–ETH Zürich; F. Braghin– Politecnico di Milano